A Layerwise Shell Stiffener and Stand-Alone Curved Beam Element

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Abstract- A two-dimensional stiffener element for laminated composite shells and plates is developed based on the Layerwise Theory of Reddy for composite laminates. The element has a displacement field compatible with that of a layerwise plate or cylindrical shell element and can be effectively employed as a stiffener for such structural elements. The element can also be used as a stand-alone curved or straight beam element. A finite element model of the theory is developed and used to validate the accuracy and suitability of this element for the bending, buckling and natural vibration analysis of stand-alone curved and straight composite laminated beams.

1. INTRODUCTION

Straight and curved beam elements made of isotropic materials and fiber reinforced composites find a wide use in the aerospace industry both as stand-alone beams and shell and plate stiffener elements. Composite laminated beam elements have been employed as stiffeners in multi-layered shells and plates to, particularly, increase the load carrying capacity and, in general, improve the overall structural response of such stiffened structures. Powerful analytical models which feature a compatible displacement pattern for both the stiffener element and the plate or shell element are required for the analysis of stiffened plates and shells. The finite element method has been used by a number of investigators to model the combined response of the stiffeners and the skin quite accurately. Kohnke and Schnobrich [1] proposed a sixteen degrees of freedom isotropic beam finite element which has a displacement field compatible with that of a cylindrical shell element from which the beam element is reduced. This element has been used successfully for analyzing eccentrically stiffened isotropic cylindrical shells. However, the application of the element is limited to stiffened shells made of isotropic materials. Venkatesh and Rao [2-3] generated a laminated stiffener element with sixteen degrees of freedom from a composite laminated anisotropic shallow thin shell element of forty eight degrees of freedom. Venkatesh and Rao later extended their formulation to parallel circle and meridional stiffener elements which exhibit a displacement field compatible with the shell skin. The stiffener element of Venkatesh and Rao is idealized as a line element with its three displacements u, v and w expressed in terms of one dimensional first order Hermite interpolation polynomials. The displacement functions of this stiffener element are, therefore, compatible with those of shell elements which use products of one-dimensional first order Hermite interpolation polynomials for the description of their displacement fields. The most serious drawback of the formulation is, however, its limitation to thin shells and its failure in capturing the effect of the transverse normal strains. On the other hand, a family of two and three-dimensional shell stiffener elements were developed by Ferguson and Clark [4] as a double degenerate of a fully three-dimensional isoparametric continuum element. These family of elements which have a transverse shear deformation capability are of a superparametric type and can also be used to model variable thickness stiffeners.

In this study, a new shell and plate stiffener element based on Reddy's Layerwise Theory [5] for composite laminates is presented. The element has both transverse shear and normal deformation capability and can be viewed as a degenerate of a layerwise shell element [6]. The displacement fields of the stiffener element are fully compatible with those of the shell element because of the use of the same expansion for the relevant displacement fields. The stand-alone or shell stiffener element may be built up of an arbitrary number of lamina with different thickness, material property and orientation. The element is assumed to exhibit a two-dimensional behavior; the two dimensions of interest being the length and the depth. A unit value is assumed for the width. The normal and shear stresses in the width direction, i.e., s_{xx}, s_{xy} and s_{xz} are neglected due to their small magnitude. However, the transverse normal stress s_{zz} is kept to maintain compatibility with the stress field of the skin material. For stiffeners and stand-alone beams made out of composite laminates, the transverse shear stresses assume a significant magnitude because of the large differences in the inplane and shear moduli of the laminate material. Hence, any refined model for such laminated composite stiffeners should have the capacity to model the transverse shear stress for accurately describing the kinematics and state of stress.

In reducing the three-dimensional constitutive equations for a generally anisotropic laminate to that of a two-dimensional beam element, emphasis is placed on eliminating the stress components s_{xx}, s_{xy} and s_{xz} only and not the corresponding strain components e_{xx}, g_{xy} and g_{xz}. The scheme for adopting this element to stiffened plates and shells is discussed in detail in Section 2.6.

2. FORMULATION

In this section, the layerwise theory for curved shell stiffener elements is presented in detail. The objective is to define the displacement field and the strain-displacement relationships and subsequently derive the equations of motion. The finite element model of the theory will be derived through the weak form of the equations of motion.

Consider a laminated curved beam element of length *L*, thickness *h*, and undeformed middle surface of radius *R*, with $h << R$, as shown in Figure 1. The derivations presented here are for the general case of a circumferential stiffener, commonly called ring, oriented in the global *y-z* plane. The choice of the global coordinate plane as *y-z* is arbitrary. These formulations will reduce to those of straight beam elements and longitudinal stiffeners if the radius of curvature is taken as infinity.

2.1 Displacement Field

The in-plane and transverse displacements *(v, w)* at a generic point *(y, z)* in the stiffener element are assumed to be of the form:

$$v(y,z) = \sum_{i=1}^{N+1} v_i(y)\ \Phi^i(z) \tag{1.a}$$

$$w(y,z) = \sum_{i=1}^{N+1} w_i(y)\ \Phi^i(z) \tag{1.b}$$

where "*N*" is the number of mathematical layers in the stiffener, v_i and w_i are the nodal values of the in-plane and transverse displacements of each interface, and F^i is a Lagrangian interpolation function through the thickness. Note that plies of identical material and geometry properties are sometimes modeled as a single "mathematical" layer to reduce computational efforts. As a result, *N* could be less than the actual total number of plies. In equation (1) and all subsequent equations, summation is implied on repeated subscripts and superscripts.

The through-the-thickness expansion of the transverse displacement "*w*" relaxes the condition for the inextensibility of the normals to the mid-surface as stipulated by the classical beam theory and the Timoshenko shear deformation theories. This allows a non-zero transverse normal strain as in the models based on elasticity theory.

Here global linear approximation functions of the following form are used:

$$\Phi^i(z) = \begin{cases} y_2^{(i-1)}(z) = \dfrac{z - z_{i-1}}{h_{i-1}}, z_{i-1} \le z \le z_i \\ y_1^{(i)}(z) = \dfrac{z_i - z}{h_i},\ z_i \le z \le z_{i+1} \end{cases} \qquad (\ i\ =\ 1, 2, \dots N\ ) \tag{2}$$

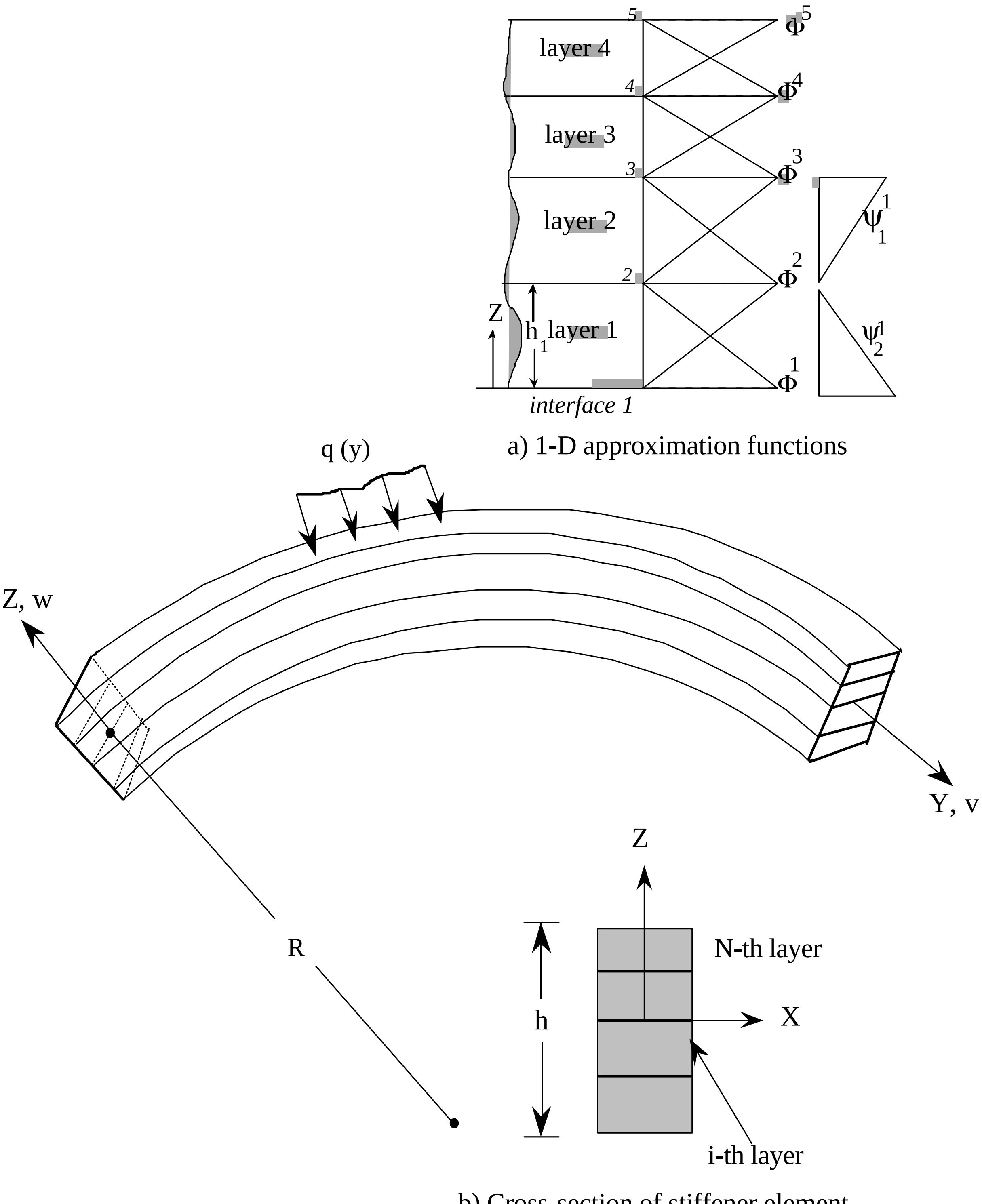

Figure 1. A Laminated Curved Shell Stiffener Element.

where ψ_j^i *(j = 1, 2)* is the local Lagrangian interpolation function associated with the *j-th* node of the i-th layer and h_i is the thickness of the *i-th* layer, as shown in Figure 1a.

Here, it could be recognized that each layer is in effect a one-dimensional finite element through the thickness. The ψ_j^i are the interpolation functions of the *i-th* element with *j = 1, 2* for linear elements and *j = 1, 2, 3* for quadratic elements.

2.2 Strain-Displacement Relationships

The von Kármán type non-linear strains are considered, where small strains and moderate rotations with respect to the curved beam reference surface are assumed. The transverse normal strain e_{zz} is also included in the formulation:

$$
\begin{aligned}
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial v_i}{\partial y} + \frac{w_i}{R}\right)\Phi^i + \frac{1}{2}\frac{\partial w_i}{\partial y}\frac{\partial w_j}{\partial y}\Phi^i\Phi^j \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \frac{v}{R} = v_i \frac{d\Phi^i}{dz} + \left(\frac{\partial w_i}{\partial y} - \frac{v_i}{R}\right)\Phi^i \qquad (3) \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = w_i \frac{d\Phi^i}{dz}
\end{aligned}
$$

2.3 Virtual Strain Energy

The virtual work due to the internal stresses in the stiffener element can be written as:

$$
dU = \int_{\Omega} \int_{-\frac{h}{2}}^{\frac{h}{2}} (s_{yy} de_{yy} + s_{zz} de_{zz} + s_{yz} dg_{yz})\, dz\, dA \qquad (4)
$$

where s_{yy}, s_{yz}, and s_{zz} are the in-plane, transverse shear and transverse normal stresses respectively. V (i.e., ∫dzdA) is the total volume of the stiffener element, and W is the area of the reference surface.

Substituting the strain-displacement relations given by equations (3) into the virtual work statement, equation (4), gives an expression for the virtual strain energy in terms of the displacements as:

$$\delta U = \int_{\Omega} [N_y^i (\frac{\partial \delta v_i}{\partial y} + \frac{\partial w_i}{R}) + \frac{1}{2} M_y^{ij} \frac{\partial \delta w_i}{\partial y} \frac{\partial w_i}{\partial y} + Q_{zz}^i \delta w_i + Q_{yz}^i \delta v_i + K_{yz}^i \frac{\partial \delta w_i}{\partial y} - \frac{1}{R} K_{yz}^i \delta v_i] \, dA \qquad (5)$$

The laminate force and moment resultants used in equation (5) are defined in terms of the internal stresses and the through-the-thickness interpolation polynomials and have the form:

$$N_y^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} \Phi^i dz \qquad M_y^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} \Phi^i \Phi^j dz \qquad (6)$$

$$K_{yz}^i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz} \Phi^i dz \qquad (Q_{yz}^i, Q_{zz}^i) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{yz}, \sigma_{zz}) \frac{d\Phi^i}{dz} dz$$

2.4 Constitutive Equations and Laminate Stiffeners

The three-dimensional constitutive equations of an anisotropic body are reduced to a two-dimensional form by eliminating the normal stress σ_{xx}, the shear stress σ_{xy}, and the transverse shear stress σ_{xz}. Here, it should be borne in mind that it is only the indicated stresses which are ignored and not the corresponding strains. Thus, for a two-dimensional beam element, the constitutive equations reduce to the form:

$$\begin{Bmatrix} \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11}^* & C_{13}^* & 0 \\ C_{13}^* & C_{33}^* & 0 \\ 0 & 0 & C_{44}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \end{Bmatrix} \qquad (7)$$

The modified elastic constants C^*_{ij} are obtained by expanding all the six components of the strain vector in terms of the transformed reduced elastic constants C_{ij} and the stress vector σ_{ij} and then eliminating ε_{xx}, γ_{xy} and γ_{xz} from the ensuing equations.

$$C_{11}^* = C_{11} + \frac{C_{16}C_{26} - C_{12}C_{66}}{C_{22}C_{66} - C_{26}C_{26}} C_{12} + \frac{C_{12}C_{26} - C_{16}C_{22}}{C_{22}C_{66} - C_{26}C_{26}} C_{16}$$

$$C_{13}^* = C_{13} + \frac{C_{33}C_{26} - C_{23}C_{66}}{C_{22}C_{66} - C_{26}C_{26}} C_{23} + \frac{C_{12}C_{26} - C_{16}C_{22}}{C_{22}C_{66} - C_{26}C_{26}} C_{36} \qquad (8)$$

$$C_{33}^* = C_{33} + \frac{C_{36}C_{26} - C_{23}C_{66}}{C_{22}C_{66} - C_{26}C_{26}} C_{23} + \frac{C_{23}C_{26} - C_{36}C_{22}}{C_{22}C_{66} - C_{26}C_{26}} C_{36}$$
$$C_{44}^* = C_{44} - \frac{C_{45}C_{45}}{C_{55}}$$

Substituting the constitutive relations given by equations (7) into the laminate force resultants of equation (6), an expression for the laminate force resultants in terms of the displacements is obtained:

$$\begin{aligned}
N_y^i &= A_{11}^{ij}\left(\frac{\partial v_j}{\partial y} + \frac{w_j}{R}\right) + \overline{A}_{13}^{ij} w_j + \frac{1}{2} D_{11}^{ijk} \frac{\partial w_j}{\partial y}\frac{\partial w_k}{\partial y} \\
Q_{zz}^i &= \overline{A}_{13}^{ji}\left(\frac{\partial v_j}{\partial y} + \frac{w_j}{R}\right) + \overline{\overline{A}}_{33}^{ij} w_j + \frac{1}{2} D_{13}^{jki} \frac{\partial w_j}{\partial y}\frac{\partial w_k}{\partial y} \\
Q_{yz}^i &= \overline{A}_{44}^{ji}\left(\frac{\partial w_j}{\partial y} - \frac{v_j}{R}\right) + \overline{\overline{A}}_{44}^{ji} v_j \\
K_{yz}^i &= A_{44}^{ij}\left(\frac{\partial w_j}{\partial y} - \frac{v_j}{R}\right) + \overline{A}_{44}^{ij} v_j \\
M_y^{ij} &= D_{11}^{ijk}\left(\frac{\partial v_k}{\partial y} + \frac{w_k}{R}\right) + \frac{1}{2} F_{11}^{ijkl} \frac{\partial w_k}{\partial y}\frac{\partial w_l}{\partial y}
\end{aligned} \tag{9}$$

where the laminate stiffness coefficients are given in terms of the modified elastic constants and the through-the-thickness interpolation polynomials as:

$$\begin{aligned}
A_{mn}^{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{mn}^* \Phi^i \Phi^j dz & \overline{A}_{mn}^{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{mn}^* \Phi^i \frac{d\Phi^j}{dz} dz \\
\overline{\overline{A}}_{mn}^{ij} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{mn}^* \frac{d\Phi^i}{dz}\frac{d\Phi^j}{dz} dz & D_{mn}^{ijk} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{mn}^* \Phi^i \Phi^j \Phi^k dz \\
\overline{D}_{mn}^{ijk} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{mn}^* \Phi^i \Phi^j \frac{d\Phi^k}{dz} dz & F_{mn}^{ijkl} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{mn}^* \Phi^i \Phi^j \Phi^k \Phi^l dz
\end{aligned} \tag{10}$$

Here $i, j = 1, 2, ..., N+1$ and $m, n = 1, 3, 4$, where N is the number of mathematical layers in the stiffener under consideration. Note that the ply stiffness coefficients with a single bar are not symmetric with respect to the superscripts. All the other stiffness coefficients are symmetric with respect to both superscripts and subscripts. The symmetry in the subscripts is, of course, due to the symmetry in the elastic constants of the material of the laminates. The definition of the integrals in equation (10) accounts for the symmetry in the superscripts for all the laminate stiffness coefficients except the ones with single bars on them.

2.5 Finite Element Model

A displacement finite element model of the governing equations defined in the previous sections is developed here. The interface displacements (v_i, w_i) are expressed over each element as a linear combination of the surface-wise one-dimensional Lagrange interpolation function φ_j and the nodal interface displacements v_i^j and w_i^j as follows:

$$(v_i, w_i) = \sum_{j=1}^{p} (v_i^j, w_i^j)\,\varphi_j \tag{11}$$

where p is the number of nodes in a typical element. The variational form of the virtual work statement, equation (5), is developed by multiplying it with a test function and integrating the product by parts. Equation (11) and (9) are then substituted into the resulting variational form and the finite element model is developed as:

$$\begin{bmatrix} {}^{11}K_{ij}^{ab} & {}^{12}K_{ij}^{ab} \\ {}^{21}K_{ij}^{ab} & {}^{22}K_{ij}^{ab} \end{bmatrix} \begin{Bmatrix} \{v\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} \{q_v\} \\ \{q_w\} \end{Bmatrix} \tag{12}$$

where *[K]* is the element stiffness matrix, *{v}* and *{w}* are the interface displacement vectors and $\{q_v\}$ and $\{q_w\}$ are the corresponding load vectors. Here $i, j = 1, 2, ..., p$ and $a, b = 1, 2, ..., N+1$. The elements of the stiffness matrix *[K]* for a shell/plate stiffener are given in Reference [6]. For eigenvalue analysis, (i.e., linearized buckling or vibration) the finite element model (12) takes the form,

$$\left(\begin{bmatrix} {}^{11}K_{ij}^{ab} & {}^{12}K_{ij}^{ab} \\ {}^{21}K_{ij}^{ab} & {}^{22}K_{ij}^{ab} \end{bmatrix} - l \begin{bmatrix} {}^{11}S_{ij}^{ab} & 0 \\ 0 & {}^{22}S_{ij}^{ab} \end{bmatrix} \right) \begin{Bmatrix} \{v\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{13}$$

For natural vibration analysis, the lowest l is the square of the fundamental frequency and *[S]* is the mass matrix, given in terms of the layer density and laminate mass coefficient as:

$$^{11}S_{ij}^{ab} \quad = \quad ^{22}S_{ij}^{ab} \quad = \quad \int_{\Omega} \boldsymbol{r}^{a} (G^{ab} \boldsymbol{j}_i \boldsymbol{j}_j) dy \tag{14a}$$

where $\boldsymbol{r}^{\boldsymbol{a}}$ is the density of the composite material of layer-$\boldsymbol{a}$. The laminate mass coefficients $G^{\boldsymbol{ab}}$ are functions of the through-the-thickness integration polynomials and are defined as:

$$G^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \Phi^i \Phi^j dz \tag{14b}$$

For a linearized stability analysis, the lowest $\boldsymbol{l}$ is the critical buckling load and *[S]* is the geometric stiffness matrix,

$$^{11}S_{ij}^{\alpha\beta} = 0 \qquad \text{and} \qquad ^{22}S_{ij}^{ab} = \int_{\Omega} (G^{ab} \frac{¶\boldsymbol{j}_i}{¶y} \frac{¶\boldsymbol{j}_j}{¶y}) dy \tag{15}$$

2.6 Modeling of Stiffened Plates and Shells

Since both the shell and stiffener elements have the same displacement fields, compatibility of strains and equilibrium of forces can be enforced by letting the two elements share the same nodes in the finite element mesh. The linear layerwise stiffener elements share one of the sides of the two-dimensional quadrilateral surface-wise shell elements. Depending on the location of the stiffener, the stiffener element will again share one or more of the through-thickness nodes of the skin element. Stiffeners attached to either the outside or inside of the skin share one node whereas stiffeners attached to both sides of the skin share two nodes through the thickness. Apart from enforcing compatibility of strains and equilibrium of forces, the sharing of nodes significantly reduces the size of the equations to be assembled and solved. Unlike analytical models which "smear" or average out the properties of the stiffeners over the surface irrespective of the location of the stiffeners, the layerwise format of the stiffener and shell skin element takes care of eccentricity automatically. The effect of the location of the stiffeners is also handled readily by the model developed. The application of the layerwise stiffener element for the analysis of stiffened composite plates and shells is discussed in detail by Kassegne and Reddy [12].

3. NUMERICAL EXAMPLES

The finite element model developed for linear and nonlinear bending, vibration and linear buckling analysis is used to investigate a number of sample problems and verify its accuracy and highlight its advantages over the conventional approaches. In this section, a number of isotropic and orthotropic single layer and laminated composite beams are analyzed for natural vibration, critical buckling load and static loads to verify the efficiency and accuracy of the element. Both curved and straight stand-alone beams are investigated.

3.1 Straight Isotropic Beam

A cantilever isotropic beam, shown in Figure 2, is analyzed under a tip point load for different length to thickness *(L/H)* ratios. The radius is set to infinity. A convergent solution is obtained for four linear Lagrange elements along the length. Through the thickness, two runs involving two and five linear elements were carried out and the results are tabulated in Table 1.

The material properties used are:

$E = 3.0x10^6\ lb/in^2\ ; \quad \mathbf{n} = 0$

As shown in Table 1, the layerwise theory gives a very satisfactory result for both deep beams and shallow beams. The results in Reference [7] were obtained using a continuum degenerate element. This particular example illustrates the efficiency of the layerwise model in capturing shear deformations in deep beams. In terms of convergence, faster rate is observed for shallow beams which required only two linear elements through the thickness. Deep beams required finer discretization through the thickness.

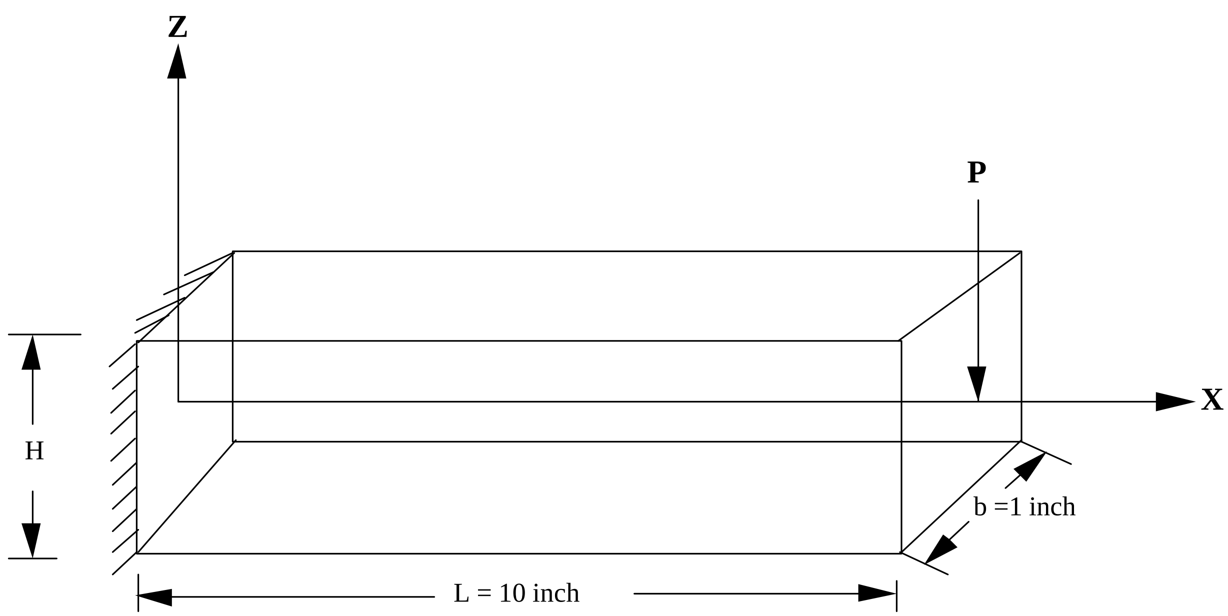

Figure 2. Straight Isotropic Beam.

Table 1. Linear displacement of a straight beam (in inches).

P in lbs	L/H	Displacement in inches		
		Layerwise Theory	Reference [7]	Classical Beam Theory
2.0	1.0	0.419E-05 (2-layers) 0.420E-05 (5-layers)	0.427E-05	0.267E-05
3.0	10.0	0.402E-02 (2-layers)	0.402E-02	0.400E-02

3.2 Simply Supported (0/90/0) Beam under Sinusoidal Load

This is a classical problem often used by researchers as a bench mark. This problem has been solved by Pagano [9] using a three-dimensional elasticity theory for the case of a plate in cylindrical bending. The intensity of the sinusoidal load is q as shown in Figure 3. The beam consists of three equal thickness graphite-epoxy laminates and has a length to depth ratio of four ($L/H = 4$).

The material properties for the Graphite/Epoxy composite laminate are as follows:

$E_{11} = 172\ GPa,\ E_{22} = E_{33} = 6.9\ GPa;$

$G_{23} = 3.4\ GPa,\ G_{12} = G_{13} = 1.4\ GPa;$

$m_{12} = 0.25,\ m_{13} = 0.25,\ m_{23} = 0.25$

A convergent solution is obtained for four quadratic elements along the length and a total of six linear elements through the thickness (i.e., two linear elements per lamina). Three-node quadratic elements with reduced integration are used since two-node elements can model only constant moments across their length. The through-thickness distribution of the longitudinal stress s_{xx} is given in Figure 4. The stresses are computed at the mid-span and are non-dimensionalized by dividing by q. The normal stresses computed by the layerwise theory agree very closely with the exact solutions, whereas CLT fails to predict accurately even the sign of the stress at the interface of the two laminates.

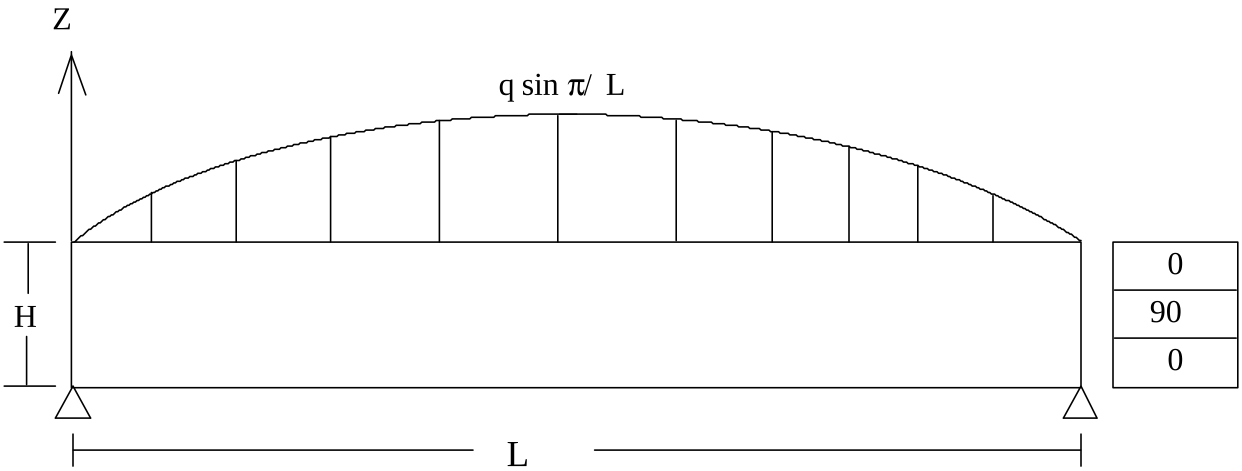

Figure 3. Simply Supported Composite Beam (0/90/0) under sinusoidal load.

3.3 Warping Behavior of (0/90/0) Composite Beam

A triple layer (0/90/0) symmetric composite beam subjected to a sinusoidal load is analyzed to investigate its warping deformations for different L/H ratios. The problem was originally solved by Bhate et al [13].

The material properties are:

$E_{11} = E_{22} = 1 .0E+06$ *psi*,

$G_{12} = G_{13} = G_{23} = 4.0E+05$ *psi*;

$m_{12} = m_{13} = m_{23} = 0.25$

The finite element mesh used for this particular beam consists of nine three-node Lagrange quadratic elements along the length and a total of six quadratic elements through the thickness. For lower values of L/H ratios (i.e., shallow beams), no noticeable warping is observed. However as the L/H ratio increases, significant warping of the cross section takes place. The warping displacements progressively increase towards the free end. Figure 6 shows these warping displacements for a (0/90/0) composite beam of L/H ratio of 10. The same results were reported by Bhate et al [13].

The ability of this layerwise approach to capture warping displacements is not unexpected since the assumption of plane sections remaining plane after deformation (as used in classical beam theories) has been removed by expanding the in-plane displacement fields through the thickness as functions of a Lagrange interpolation functions. Here, it is interesting to note that the use of quadratic elements depth-wise not only achieves faster convergence but also manages to capture the non-linear through-the-thickness variation of the in-plane displacements. A significantly higher number of linear Lagrange elements will be required through the depth of the beam to capture the same warping displacements.

3.4 Natural Vibration of Orthotropic and Cross-Ply Beams

Simply supported orthotropic and cross-ply beams made of graphite/epoxy material are investigated to determine their fundamental frequencies using the layerwise theory for beams. The results are given in a non-dimensionalized form as :

$$\Omega^2 = \omega^2 \frac{h^2 E_2}{\rho L^4 (1 - \mu_{23}\mu_{32})}$$

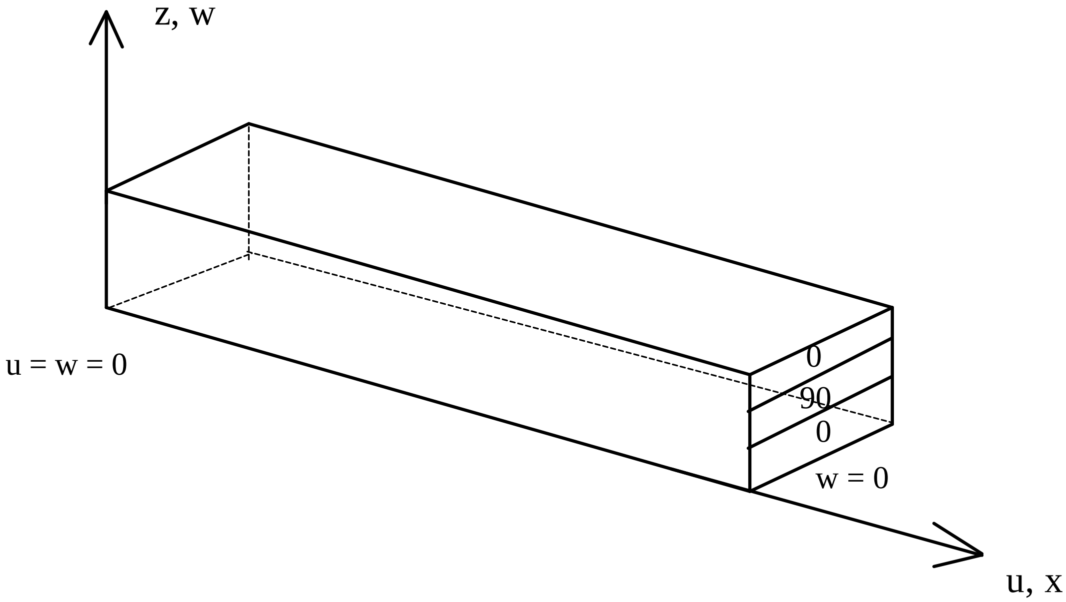

Figure 5. Simply supported composite beam under sinusoidal load.

The material properties for the Graphite/Epoxy composite laminate are as follows:

E_{11} = 181 GPa , E_{22} = E_{33} = 10.3 GPa;
G_{23} = 6.21GPa , G_{12} = G_{13} = 7.17 GPa;
m_{12} = 0.28, m_{13} = 0.02, m_{23} = 0.4, r = 1389.23 Kg/m^3

The effect of ply orientation on the frequencies is demonstrated in Table 2. Convergent results were obtained for eight linear elements along the length and two linear elements through the thickness. Note that the Euler-Bernoulli theory for beams is inadequate in capturing the effect of the ply orientation. The layerwise theory, unlike the classical beam theory, does not neglect the out-of-plane normal and transverse shear strains. As a result, these quantities are accounted for in the constitutive equations. This explains the capacity of the present model to capture the effect ply orientation has on the response of the laminate. Note that Reference [8] uses a parabolic shear deformation beam theory.

Table 2. Comparison of natural frequencies (Ω) of a composite beam.

Lamination Scheme	Layerwise Theory	Reference [8]
Isotropic	2.52	2.67
Orthotropic	10.8	10.62
0/90/0	10.0	10.40

Table 3. Effect of ply orientation on the natural frequency Ω.

Angle (θ)	0^{o}	15^{o}	75^{o}
Layerwise Theory (3 linear elements)	10.8	7.42	2.92
Reference [8]	10.62	7.31	2.87
Euler-Bernoulli	11.89	11.11	3.05

3.4 Buckling of Curved and Straight Beams

A linearized buckling analysis of straight and curved isotropic beams under different boundary conditions is carried out. The results are compared with those reported by Rao and Tripathy [10] where a sixteen degrees-of-freedom line element based on the classical theory is used. The beams are modeled with two linear elements through the thickness and eight linear elements along the length.

The material and geometrical properties used are:

$E = 2.0x10^6\ lb/in^2;\ \nu = 0.3$

$H = 25.4\ mm;\ Width = 25.4\ mm\ \ and\ \ L = 508\ mm$

Table 11. Critical loads of a straight beam under different boundary conditions.

Boundary Condition	Layerwise Theory	Reference [10]
Simply Supported	94.43 kN	96.6 kN
Clamped-Free	23.15 kN	24.0 kN

Table 12. Critical loads of a curved beam under uniform radial load (simply supported).

No of Elements	Layerwise Theory	Reference [10]
2	78.54 kN/m	78.76 kN/m
6	76.87 kN/m	77.05 kN/m

4. CONCLUSIONS

A two-dimensional layerwise element which may be used as a stand-alone beam or as a stiffener for laminated composite shells and plates is developed based on Reddy's layerwise theory for composite laminates. With its transverse shear and transverse normal deformation modeling capability, the stiffener element developed here exhibits a displacement field compatible with that of a cylindrical shell layerwise element [6, 11]. The element is tested for perfomance in predicting stress magnitudes and distribution, complex deformations such as warping, critical buckling loads and natural vibration for a good variety of beam structures. The results obtained from the finite element model developed agree very well with those reported in the literature. The model is observed to exhibit a high accuracy in determining quantities of importance for composite beams and stiffener elements, such as through-the-thickness distribution of stresses and strains, shear deformations and warping displacements.

5. REFERENCES

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